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Inequivalent, bordant group actions on a surface

BY CHARLES LIVINGSTON
Department of Mathematics, Indiana University, Bloomington, IN 47405, *U.S.A.*

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An action of a group, G, on a surface, F, consists of a homomorphism

$$\phi: G \to \text{Homeo}\,(F).$$

We will restrict our discussion to finite groups acting on closed, connected, orientable surfaces, with $\phi(g)$ orientation-preserving for all $g \in G$. In addition we will consider only effective (ϕ is injective) free actions. *Free* means that $\phi(g)$ is fixed-point-free for all $g \in G$, $g \neq 1$. This paper addresses the classification of such actions.

Given an action of G on F, as above, new actions can be constructed by conjugating with homeomorphisms of F. Hence, two actions, ϕ_1 and ϕ_2, are called equivalent if there is a homeomorphism h of F such that $\phi_2(g) = h \circ \phi_1(g) \circ h^{-1}$, for all $g \in G$.

One invariant of equivalence classes of actions is their free G-bordism class. Essentially, G actions on surfaces F_1 and F_2 are freely G-bordant if there is a compact, orientable 3-manifold, M, with boundary $F_1 \cup -F_2$, such that there is a free orientation-preserving action of G on M restricting to the two given actions on the boundary components of M. (See [**2**] for details.) The set of G-bordism classes forms a group, denoted $\theta_2^{\text{free}}(G)$, which is isomorphic to $H_2(G)$.

It follows from the definitions that equivalent actions are freely bordant. The point of this paper is that the converse does not hold. We will construct an example of two inequivalent actions of the symmetric group, S_8, on a surface such that both actions are null bordant.

Although it seemed unlikely that bordism could classify group actions, partial results in this direction have been obtained. In particular, freely bordant actions are equivalent in the case that G is cyclic [**6**], abelian [**3**], or metacyclic [**4**]. Related results have also been proved for the unoriented case [**8**, **9**]. Furthermore, freely bordant actions are equivalent in a fairly strong stable sense [**5**].

Thanks are due here to Allan Edmonds, for pointing out this problem to me and for many valuable conversations, as well as to Charlie Frohman and Richard Skora.

1. *Group representations and an outline of the construction*

In order to construct our example we will take advantage of the link between group actions and surjective representations of surface groups to finite groups. This relationship is provided by covering-space theory and is described in detail in [**3**]. A summary follows.

Two surjective representations, ρ and η of $\pi_1(F, x_0)$ to G are called *equivalent* if there is a homeomorphism, h, of (F, x_0) such that $\rho \circ h_* = \eta$. Two such representations are called *bordant* if they determine the same classes in the bordism group $\Omega_2(K(G, 1))$.

There is an exact relationship between equivalence and free bordism of group actions and equivalence and bordism of representations, as described in [**3**].

In the present situation it suffices to construct two inequivalent surjective representations of a surface group which are bordant.

Outline of the construction

We will construct two surjective representations, ρ and η, of the fundamental group of a genus 2 surface, F_2, to the symmetric group, S_8. Both will extend to a solid handlebody and hence will be null bordant.

The representation ρ will be described in a roundabout way. F_2 is the 2-fold branched cover of the 2-sphere, S^2, branched over 6 points. Certain representations of the fundamental group of the 6-times punctured 2-sphere can be lifted to $\pi_1(F_2)$. ρ will arise in this way.

The reason for using this procedure for constructing ρ is the following fact. Any homeomorphism of F_2 is isotopic to a homeomorphism which is the lift of a homeomorphism of S^2 via the 2-fold branched covering map [**1**]. Essentially, this lets us study F_2 (and ρ) in terms of S^2.

The representation η will be described explicitly in terms of its values on a standard generating set for $\pi_1(F_2)$.

The representations will be distinguished by their behaviour on simple closed curves on F_2. Now a simple closed curve on a surface does not determine an element of the fundamental group of that surface, because of basepoint considerations, but only a conjugacy class in the fundamental group. Hence, for any representation $\nu\colon \pi_1(F) \to G$, each simple closed curve γ on F determines a conjugacy class in G. It depends only on the homotopy class of γ.

Finally, our calculation will be simplified by the fact that in S_8 the conjugacy class of an element is determined by its type. The *type* of an element is a description of its cyclic decomposition. For instance, a transposition is of type (2), the product of a 3-cycle and a disjoint 5-cycle is (3)(5). Two elements are conjugate if and only if they are of the same type.

2. *Details of the construction*

Notation. Let $p\colon F_2 \to S^2$ be the 2-fold branched covering of S^2, branched over the points $q_1, \ldots, q_6 \in S^2$. Set $p^{-1}(q_i) = \tilde{q}_i \in F_2$ for $i = 1, \ldots, 6$. Denote by Q the set $\{q_1, \ldots, q_6\}$ and by $\tilde{Q}$ the set $\{\tilde{q}_1, \ldots, \tilde{q}_6\}$. Pick a basepoint x_0 for $\pi_1(S^2 - Q)$ and $\tilde{x}_0$ for $\pi_1(F_2 - \tilde{Q})$ such that $p(x_0) = x_0$.

$\pi_1(S^2 - Q)$ is generated by elements $c_1, \ldots, c_6$. $\pi_1(F_2 - \tilde{Q})$ is generated by elements $a_1, b_1, a_2, b_2, c_1', \ldots, c_6'$. All these elements and the branched cover are illustrated in Figure 1, except for the c_i'. c_i' is represented by a path on F_2 running from $\tilde{x}_0$ to a point near $\tilde{q}_i$, once around $\tilde{q}_i$, and back to $\tilde{x}_0$. We have the following:

$$p_*(a_1) = c_1 c_2, \quad p_*(b_1) = c_2 c_3, \quad p_*(a_2) = c_5 c_6, \quad p_*(b_2) = c_4 c_5 \quad \text{and} \quad p_*(c_i') = c_i^2,$$

The inclusion $f\colon (F_2 - \tilde{Q}) \to F_2$ induces a map f_* on $\pi_1(F_2 - Q)$ with the property that $f_*(c_i') = 1$ for $i = 1, \ldots, 6$, and the four elements $f_*(a_1)$, $f_*(b_1)$, $f_*(a_2)$, and $f_*(b_2)$ form a standard basis for $\pi_1(F_2)$. We denote these elements by a_1, b_1, a_2, b_2 as well, when no confusion will result.

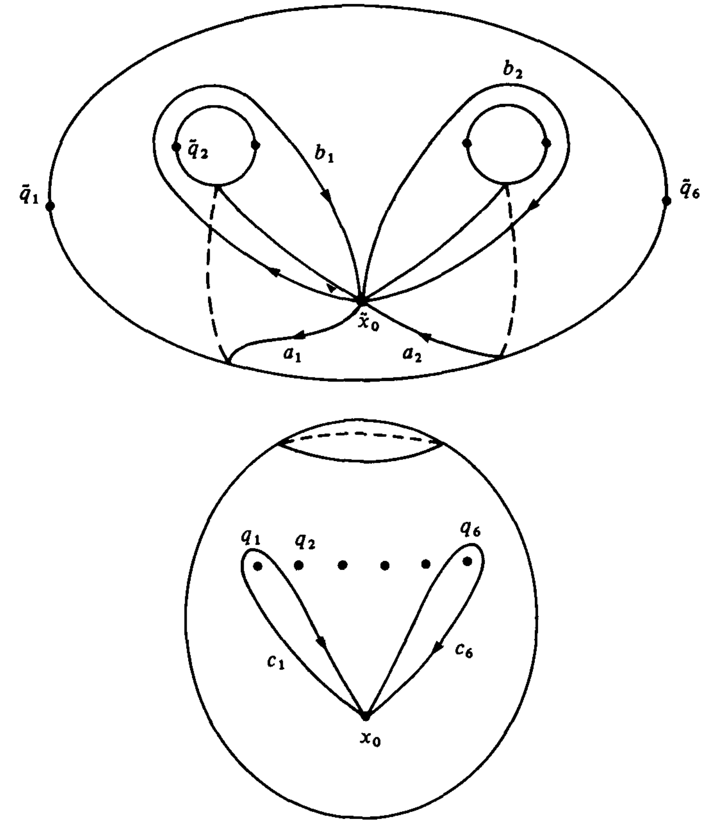

Fig. 1.

Construction of ρ

Consider the representation $\nu: \pi_1(S^2 - Q) \to S_8$ defined by

$$\nu(c_1) = \nu(c_2) = (12)(34)(56)(78) = \tau_1,$$
$$\nu(c_3) = \nu(c_4) = (23)(45)(61)(78) = \tau_2,$$
$$\nu(c_5) = \nu(c_6) = (17)(24)(38) = \tau_3.$$

The composition $\nu \circ p_*$ defines a representation of $\pi_1(F_2 - \tilde{Q})$ to S_8. The kernel of the inclusion $\pi_1(F_2 - \tilde{Q}) \to \pi_1(F_2)$ is normally generated by the set $\{c_i', \ldots, c_6'\}$. $\nu \circ p_*$ vanishes on this kernel, since $\nu \circ p_*(c_i') = \nu(c_i^2) = \tau_j^2 = 1$ for some j. Hence $\nu \circ p_*$ factors through a representation $\rho: \pi_1(F_2) \to S_8$.

We must show that ρ is surjective. This will follow from a demonstration that ν is surjective, for the following reason. Since p restricted to $F_2 - \tilde{Q}$ is a 2-fold covering of $S^2 - Q$, the image of $\pi_1(F_2)$ under ρ is either onto the image of ν, or else of index 2 in the image of ν. Given that ν is onto S_8, we have that ρ is either onto S_8 or A_8. Note that $\rho(b_2) = \nu(c_5 c_6) = \tau_2 \tau_3 \notin A_8$. Hence ρ is onto.

To complete the argument we need to show that ν is onto. First note that

$$(\tau_1 \tau_3)^3 = (56).$$

Secondly one can show that the subgroup of S_8 generated by τ_1, τ_2, and τ_3 acts transitively on the set of transpositions in S_8 by conjugation. There are only

28 transpositions in S_8, and this calculation is easily done by hand. We leave it to the reader. Hence, the image of ν contains all transpositions, and is therefore onto.

Finally, observe that ρ is given explicitly by

$$\begin{aligned}\rho(a_1) &= \nu(c_1c_2) = \tau_1^2 = 1\\ \rho(b_1) &= \nu(c_2c_3) = \tau_1\tau_2 = (135)(264)\\ \rho(a_2) &= \nu(c_5c_6) = \tau_3^2 = 1\\ \rho(b_2) &= \nu(c_4c_5) = \tau_2\tau_3 = (16734528).\end{aligned}$$

Since $\rho(a_1) = \rho(a_2) = 1$, ρ extends to a representation of the fundamental group of a solid handlebody in the obvious way, and is therefore null bordant.

Description of η

The representation $\eta\colon \pi_1(F_2) \to S_8$ is defined by

$$\begin{aligned}\eta(a_1) &= \eta(a_2) = 1,\\ \eta(b_1) &= (12),\\ \eta(b_2) &= (2345678).\end{aligned}$$

Notice that η is null bordant.

ρ and η are inequivalent

Given a representation $\nu\colon \pi_1(F_2) \to S_8$ and a closed path γ in F_2, denote by $T_\nu(\gamma)$ the type of $\nu(\gamma)$ in S_8. Recall from Section 1 that this is well defined and depends only on the (free) homotopy class of γ (as well as ν, of course).

Observe that there are two disjoint simple closed curves on F_2, γ_1 and γ_2, with the properties that $T_\eta(\gamma_1) = (2)$, $T_\eta(\gamma_2) = (7)$, and $F_2 - (\gamma_1 \cup \gamma_2)$ is connected. For instance, γ_1 can be freely homotopic to b_1, and γ_2 can be freely homotopic to b_2. Clearly, if ρ were equivalent to η a similar pair of curves could be found, with ρ replacing η. The argument is completed with the following proposition.

PROPOSITION 1. *If γ_1 and γ_2 are disjoint simple closed curves on F_2 such that*

$$F_2 - (\gamma_1 \cup \gamma_2)$$

is connected, then either $T_\rho(\gamma_1) \neq (2)$ or $T_\rho(\gamma_2) \neq (7)$.

The proof of this proposition occupies the next section.

3. *Proof of Proposition* 1

Suppose that such a pair of curves, γ_1 and γ_2, existed such that $T_\rho(\gamma_1) = (2)$ and $T_\rho(\gamma_2) = (7)$.

Step 1. There exists a second pair, γ_1' and γ_2', isotopic to γ_1 and γ_2, with the additional property that p (the branched covering map) maps $\gamma_1' \cup \gamma_2'$ homeomorphically onto its image, and $(\gamma_1' \cup \gamma_2') \cap \tilde{Q} = \varnothing$. (Equivalently $(\gamma_1' \cup \gamma_2') \cap g(\gamma_1' \cup \gamma_2') = \varnothing$, where g is the branched covering translation.)

To prove this, first pick a pair of disjoint simple closed curves, ω_1 and ω_2, such that $F_2 - (\omega_1 \cup \omega_2)$ is connected and p, restricted to $\omega_1 \cup \omega_2$, is a homeomorphism onto its image, with $(w_1 \cup w_2) \cap \tilde{Q} = \varnothing$. Such a pair is easily constructed. For instance, let λ_1 be the boundary of a disc on S^2 containing q_1 and q_2, and let λ_2 be the boundary of a disjoint disc on S_2 containing q_5 and q_6. Set ω_i to be one of the two components of $p^{-1}(\lambda_i)$, for $i = 1$ and 2. In this case ω_1 and ω_2 will be meridians homotopic to a_1 and a_2.

There is a homeomorphism, h, of F_2 such that $h(\omega_1) = \gamma_1$ and $h(\omega_2) = \gamma_2$. This is an easy exercise, using the classification of surfaces, applied to show that $F_2 - (\omega_1 \cup \omega_2)$ is homeomorphic to $F_2 - (\gamma_1 \cup \gamma_2)$.

Now h is isotopic to a second homeomorphism, h', such that h' is the lift of a homeomorphism $\bar{h}'$ of S^2 via p. That is, h' and $\bar{h}'$ satisfy the relationship $\bar{h}' \circ p = p \circ h'$. Furthermore, the sets Q and $\tilde{Q}$ are invariant under $\bar{h}'$ and h', respectively. (See [**1**], chapter 4 for details.)

Set $\gamma_1' = h'(\omega_1)$ and $\gamma_2' = h'(\omega_2)$. Clearly γ_1' and γ_2' are isotopic to γ_1 and γ_2 respectively. The fact that p maps $\gamma_1' \cup \gamma_2'$ homeomorphically onto its image follows from the similar statement for $\omega_1 \cup \omega_2$ and that both h' and $\bar{h}'$ are homeomorphisms.

Step 2. $p(\gamma_1')$ and $p(\gamma_2')$ bound disjoint discs, D_1 and D_2, on S^2 such that D_1 and D_2 each contain exactly 2 of the q_i:

If, in Step 1, the explicit pair ω_1 and ω_2 described in the second paragraph is used, this statement about $p(\gamma_1')$ and $p(\gamma_2')$ follows from a similar statement for $p(\omega_1)$ and $p(\omega_2)$. Although this suffices for our needs, we will outline the more general argument.

The condition that γ_i' maps homeomorphically onto its image implies that each D_i contains an even number of the q_i. Each D_i must contain at least 2 of the q_i, or else one of the γ_i' would disconnect F_2. Finally, if one of the D_i contained 2 of the q_i and the other D_i contained 4 of the q_i, then the boundaries of the D_i would be homotopic, and $F_2 - (\gamma_1' \cup \gamma_2')$ would be disconnected.

Step 3. It remains to analyse $T_\rho(\gamma_1')$ and $T_\rho(\gamma_2')$.

Since γ_1' and γ_2' are disjoint from $\tilde{Q}$, $T_\rho(\gamma_i') = T_\nu(p(\gamma_i'))$ for $i = 1$ and 2. Here ν is the representation of $\pi_1(S^2 - Q)$ to S_8 used in Section 2 to define ρ.

Pick paths joining $p(\gamma_1')$ and $p(\gamma_2')$ to the basepoint of $\pi_1(S^2 - Q)$. Denote the elements of $\pi_1(S^2 - Q)$ so determined by β_1 and β_2. We will now study $\nu(\beta_1)$ and $\nu(\beta_2)$, elements of S_8.

It follows form Step 2 that $\beta_1 = \alpha_1\alpha_2$ and $\beta_2 = \sigma_1\sigma_2$, where each α_i and σ_i is the conjugate of some c_i. In addition, the four c_i arising in this way are distinct.

For convenience, we recall that $\nu(c_i)$ if of type (2)(2)(2)(2) if $i = 1, 2, 3$, or 4, and $\nu(c_i)$ is of type (2)(2)(2) if $i = 5$ or 6. We are assuming that $\nu(\beta_1)$ is of type (2) and $\nu(\beta_2)$ is of type (7).

Since $\nu(\beta_1)$ is assumed to be an odd permutation, either $\nu(\alpha_1)$ or $\nu(\alpha_2)$ is of type (2)(2)(2) and the other is of type (2)(2)(2)(2). Since $\nu(\beta_2)$ is even, either both $\nu(\sigma_1)$ and $\nu(\sigma_2)$ are of type (2)(2)(2)(2) or of type (2)(2)(2). However, since for only two of the c_i is $\nu(c_i)$ of type (2)(2)(2), and one of the α_i is conjugate to one of these c_i, both $\nu(\sigma_1)$ and $\nu(\sigma_2)$ are of type (2)(2)(2)(2).

The final contradiction is achieved by noting that a 7-cycle in S_8 cannot be written as the product of 2 elements, each of type (2)(2)(2)(2). For instance, suppose the 7-cycle $(2345678) = x$ could be written as $y \cdot z$, where both y and z are of type (2)(2)(2)(2). y would contain the transposition $(1\,a)$ for some a, $2 \leqslant a \leqslant 8$. Since x fixes 1, z would have to contain the transposition $(a\,1)$. Hence $y \cdot z$ would fix a as well as 1, and hence not be a 7-cycle.

4. *Comments*

A weaker form of equivalence of group actions can be defined as follows. Actions, ϕ_1 and ϕ_2, of a group G on a surface F are *weakly equivalent* if there is a homeomorphism h of F and an automorphism k of G such that $\phi_2(g) = h \circ \phi_1(k(g)) \circ h^{-1}$ for all $g \in G$. The two representations, ρ and η, constructed above are in fact not weakly equivalent. This follows from the above arguments as well as the observation that any automorphism of S_8 preserves the types of elements. This last comment is a consequence of the fact that every automorphism of S_8 is inner. (See [7] for example.)

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